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Some notes on graphs whose spectral radius is close to $\frac{3}{2}\sqrt{2}$ ☆

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Abstract

In Belardo et al. [F. Belardo, E.M. Li Marzi, S.K. Simić, Some notes on graphs whose index is close to 2, Linear Algebra Appl. 423 (2007) 81–89] the authors considered two classes of graphs: (i) trees of order N and diameter $d = N - 3$ and (ii) unicyclic graphs of order N and girth $g = N - 2$; by assuming that each graph within these classes has two vertices of degree 3 at distance k, they order by the spectral radius the graphs from (i) for any fixed $k (1 \leqslant k \leqslant d - 2)$ and the graphs from (ii) for $1 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$. In this paper we consider two classes of graphs denoted by $P_{i,j}^{m,m,n}$ (or simply $P_{i,j}^{n}$) and $C_{s,t}^{g,k}$, containing respectively the classes (i) and (ii). The graphs in the first class ($P_{i,j}^{m,m,n}$) are paths of length n (called main paths) having two hanging paths of length m at vertices i and j. The graphs in the second class ($C_{s,t}^{g,k}$) are cycles of girth g having two hanging paths of length s and t at vertices at distance k ($1 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$). For graphs in these latter two classes we give an ordering, with respect to the spectral radius, extending the one shown in Belardo et al. (2007). Furthermore we give an upper bound for the spectral radius of the graphs in $P_{i,j}^{m,m,n}$ and a lower

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and an upper bound for graphs in $C_{s,t}^{g,k}$, following the limit point technique used in Belardo et al. [F. Belardo, E.M. Li Marzi, S.K. Simić, Path-like graphs ordered by the index, Int. J. Algebra 1 (3) (2007) 113–128].

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1. Introduction

For the basic notions and terminology on spectral graph theory we refer the readers to [5]. To make the paper more self-contained we give here just a few facts. We consider simple graphs. The characteristic polynomial of the adjacency matrix A of a graph G is called the *characteristic polynomial* of G and is denoted by $\phi(G,\lambda)(=\det(\lambda I - A))$ or simply $\phi(G)$. Its zeroes $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are real since A is symmetric and are called the eigenvalues of G. In the case of connected graphs, the largest eigenvalue λ_1 (often called *spectral radius* or *index* of G) is a simple root of $\phi(G)$ and it is here denoted by $\rho(G)$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be the Perron eigenvector of G (i.e. a positive eigenvector associated to the spectral radius, not necessarily a unit one); x_i is also called the weight of the ith vertex (with respect to $\mathbf{x}$). Then we have

$$\rho x_i = \sum_{j\sim i} x_j, \tag{1}$$

where $\sim$ denotes the adjacency relation. Eq. (1) is called the *eigenvalue equation* ($i = 1, 2, \ldots, n$) for the ith vertex (corresponding to the spectral radius).

We now introduce more notations. Let $H \subset G$ denote that H is a proper subgraph of G. Let T_{l_1,l_2,l_3} be the *starlike tree* with exactly one vertex v of degree 3, and $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where P_{l_i} is the path of order l_i ($i = 1, 2, 3$). Let us define $P_{n_1,n_2,\ldots,n_t}^{m_1,m_2,\ldots,m_t,n}$ as a path P_{n+1} of order $n+1$ ($0 \sim 1 \sim 2 \sim \cdots \sim n$) with pendant path of size m_j at vertex n_i ($i = 1, 2, \ldots, t$) whose special case $P_{i,j}^{m,m,n}$ (see Fig. 1) will be studied in this paper.

It is the spectral radius of graphs that is one of the most attracting study fields in the theory of graph spectra, and numerous papers related to it have been published (see [5] for instance). It is worth pointing out here that in [3,4] the authors determined all the graphs whose index is in the interval $(2, \sqrt{2+\sqrt{5}})$, and in [9] the authors examined the structure of graphs with index in $(\sqrt{2+\sqrt{5}}, 2/3\sqrt{2})$, where a large subcase of the graphs T_{l_1,l_2,l_3} and $P_{n_1,n_2,\ldots,n_t}^{m_1,m_2,\ldots,m_t,n}$ is included.

Recently, Belardo et al. [1] considered the index of the trees $P_{i,j}^{1,1,n-3}$ (caterpillars of diameter $d = n-3$, and denoted by $M_{i,j}^d$ therein). They gave the ordering with respect to the index

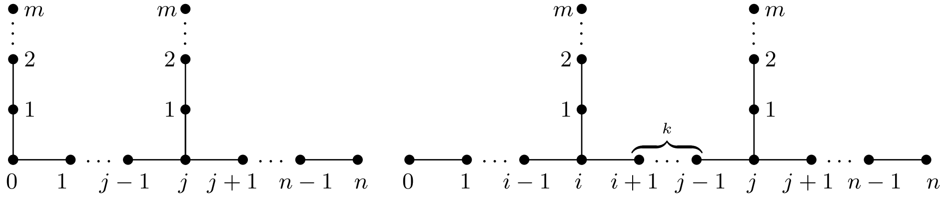

Fig. 1. The graphs $P_{0,j}^n$ and $P_{i,j}^n$.

(provided that $j - i$ is fixed) of these trees and wrote the paper in a brief way adopted in this paper.

In our paper, we will investigate the case $P_{n_i,n_j}^{m,m,n}$ (or simply $P_{i,j}^n$), where $m \geqslant 1$. So the results from [1] are here extended. One remarkable fact is that in [6] van Dam and Kooij considered the graphs $P_{n_i,n_j}^{m_i,m_j,p}$ when looking for which connected graphs on n vertices and a given diameter d the minimum spectral radius is achived, since such graphs are more resistant to virus propagation in networks. One interesting conjecture in [6] is that the graph $P_{\lfloor\frac{e-1}{2}\rfloor,n-e-\lceil\frac{e-1}{2}\rceil}^{\lfloor\frac{e-1}{2}\rfloor,\lceil\frac{e-1}{2}\rceil,n-e}$ has minimal spectral radius among the graphs on n vertices and diameter $d = n - e$ for fixed e and n large enough. We hope that our results will be helpful to solve the above conjecture.

The graphs used in the paper are shown below:

Remark 1.1. Let the distance between the vertices i and j be $D(i, j) = k + 1$. Due to symmetry we can always suppose that $i \leqslant \lfloor\frac{n-k-1}{2}\rfloor$. Write $\mathscr{G}_{i,j}^n = \{P_{i,j}^n | D(i, j) = k + 1\}$. For convenience, the path $P_{n+1}(0 \sim 1 \sim 2 \sim \cdots \sim n)$ is said to be the main path of the above graphs. Let $l(G)$ stand for the length of main path of $G \in \mathscr{G}_{i,j}^n$. Note that $P_{0,j}^n = T_{m,n-j,m+j}$.

The paper is organized as follows. In Section 2 we give some basic tools to be used through the paper. In Section 3 we prove our main results (Theorems 3.1 and 3.2), i.e. the ordering of trees $P_{i,j}^n$ with respect to the spectral radius (for k fixed); in addition, we give an upper bound for the spectral radius of the graph of this type using the limit point technique in a similar way as in [2]. Finally in Section 4 we consider a subclass of unicyclic graphs obtained from the graphs $P_{i,j}^{s,t,n}$ by adding an extra edge between the end vertices of the main path; we completely order these graphs when the lengths s and t are fixed and by using the limit point technique we also determine a lower and an upper bound for the spectral radius of the graphs considered in that section.

2. Some basic tools

The following two formulas are due to Schwenk [8] or see [5]:

Lemma 2.1 [8]. *Let $\mathscr{S}(v)(\mathscr{S}(e))$ be the set of all cycles of a graph G containing vertex v(resp. edge $e = uv$). Then*

$$\phi(G) = \lambda\phi(G - v) - \sum_{w\sim v}\phi(G - v - w) - 2\sum_{C\in\mathscr{S}(v)}\phi(G - V(C)); \tag{2}$$

$$\phi(G) = \phi(G - e) - \phi(G - u - v) - 2\sum_{C\in\mathscr{S}(e)}\phi(G - V(C)). \tag{3}$$

We assume that $\phi(G) = 1$ if G is the empty graph (i.e. with zero vertices).

Corollary 2.1. *If there are no cycles containing v and e then the formulas in Lemma 2.1 can be simplified*:

$$\phi(G) = \lambda\phi(G - v) - \sum_{w\sim v}\phi(G - v - w), \tag{4}$$

$$\phi(G) = \phi(G - e) - \phi(G - u - v). \tag{5}$$

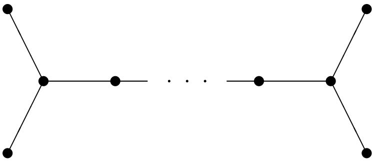

Fig. 2. The graph $W_n (n \geqslant 6)$.

Lemma 2.2 [5]. *The index of a path P_n of order n is $2\cos\frac{\pi}{n+1}$ (so less than 2).*

Lemma 2.3 [5]. *For a connected graph G and $H \subset G$, we have $\rho(H) < \rho(G)$.*

Hoffman and Smith in [7] defined an *internal path* of G as a walk $v_0v_1, \ldots, v_k (k \geqslant 1)$ such that the vertices $v_1, \ldots, v_k$ are distinct (v_0, v_k do not need to be distinct), $d(v_0) > 2$, $d(v_k) > 2$ and $d(v_i) = 2$ whenever $0 < i < k$, where $d(v)$ is the degree of the vertex v in G. They prove the following lemma:

Lemma 2.4 [7]. *Let uv be an edge of the connected graph G and let G_{uv} be obtained from G by subdividing the edge uv of G. Let $C_n(W_n, n \geqslant 6)$ be the cycle (resp. the double-snake, see Fig. 2) on n vertices.*

(i) *If uv is not in an internal path of G and if $G \neq C_n$, then $\rho(G_{uv}) > \rho(G)$.*
(ii) *If uv belongs to an internal path of G and $G \neq W_n$, then $\rho(G_{uv}) < \rho(G)$.*

The following lemma is also considered in [1], but in a slightly different form:

Lemma 2.5. *Let G and H be two graphs.*

(i) *If $\phi(G) > \phi(H)$ for any $\lambda \geqslant \rho(G)$, then $\rho(G) < \rho(H)$.*
(ii) *If $\phi(G) < \phi(H)$ for any $\lambda \in (a, b)$ and $a < \rho(G), \rho(H) < b$, then $\rho(G) > \rho(H)$.*

3. Main results

Recall, $P^n_{i,j} = P^{m,m,n}_{i,j}$.

Theorem 3.1. *If $n = 2k$, then $\rho(P^n_{i,i+k+1}) = \rho(P^n_{j,j+k+1})$ for $0 \leqslant i, j \leqslant \lfloor\frac{n-k-1}{2}\rfloor$.*

Proof. Without loss of generality, let $j = i + 1$ (with $i + 1 \leqslant \lfloor\frac{n-k-1}{2}\rfloor$). Then $k - i - 2 > 0$. Namely, since $i + 1 \leqslant \lfloor\frac{n-k-1}{2}\rfloor$ and $n = 2k$, we get $n - (k + 1) - (i + 1) = k - i - 2 \geqslant i + 1 > 0$, as claimed. We now distinguish the following two cases:

Case 1. $i = 0$.

Applying Corollary 2.1 at vertex k in $P^n_{0,k+1} = T_{m,k+m+1,k-1}$ we get

$$\begin{aligned}\phi(P^n_{0,k+1}) &= \lambda\phi^2(P_{m+k}) - \phi(P_{m+k-1})\phi(P_{m+k}) - \phi(P_{m+k})\phi(P_m)\phi(P_{k-1})\\ &= \phi(P_{m+k})[\lambda\phi(P_{m+k}) - \phi(P_{m+k-1}) - \phi(P_m)\phi(P_{k-1})]. \qquad (6)\end{aligned}$$

In the same way, for $P^n_{1,k+2}$ we get

$$\begin{aligned}\phi(P^n_{1,k+2}) &= \lambda\phi^2(T_{m,1,k-2}) - \phi(T_{m,1,k-3})\phi(T_{m,1,k-2}) - \phi(T_{m,1,k-2})\phi(P_{m+k-1}) \\ &= \phi(T_{m,1,k-2})[\lambda\phi(T_{m,1,k-2}) - \phi(T_{m,1,k-3}) - \phi(P_{m+k-1})] \\ &= \phi(T_{m,1,k-2})[\phi(T_{m,1,k-1}) - \phi(P_{m+k-1})].\end{aligned}$$

Applying Corollary 2.1 to the edge $e = uv$ in $T_{m,1,k-1}$ satisfying $d(u) = 1$ and $d(v) = 3$, we obtain $\phi(T_{m,1,k-1}) = \lambda\phi(P_{m+k}) - \phi(P_m)\phi(P_{k-1})$, and therefrom we easily get that

$$\phi(P^n_{1,k+2}) = \phi(T_{m,1,k-2})[\lambda\phi(P_{m+k}) - \phi(P_{m+k-1}) - \phi(P_m)\phi(P_{k-1})]. \tag{7}$$

Note that the spectral radius of $P^n_{0,k+1}$ and of $P^n_{1,k+2}$ is the largest root of the second factor in (6) and (7), respectively (since $P_{m+k} \subset P^n_{0,k+1}$ and $T_{m,1,k-2} \subset P^n_{1,k+2}$, see Lemma 2.3). Since these two factors are the same, we are done.

Case 2. $i \geqslant 1$.

We now apply Corollary 2.1 at vertex k in $P^n_{i,i+k+1}$. Then we arrive at

$$\begin{aligned}\phi(P^n_{i,i+k+1}) &= \lambda\phi^2(T_{m,i,k-i-1}) - \phi(T_{m,i,k-i-2})\phi(T_{m,i,k-i-1}) \\ &\quad - \phi(T_{m,i,k-i-1})\phi(T_{m,i-1,k-i-1}) \\ &= \phi(T_{m,i,k-i-1})[\lambda\phi(T_{m,i,k-i-1}) - \phi(T_{m,i,k-i-2}) - \phi(T_{m,i-1,k-i-1})].\end{aligned}$$

Since $\phi(T_{m,i+1,k-i-1}) = \lambda\phi(T_{m,i,k-i-1}) - \phi(T_{m,i-1,k-i-1})$, we have

$$\phi(P^n_{i,i+k+1}) = \phi(T_{m,i,k-i-1})[\phi(T_{m,i+1,k-i-1}) - \phi(T_{m,i,k-i-2})]. \tag{8}$$

In the same way, for $P^n_{i+1,i+k+2}$ we get that

$$\phi(P^n_{i+1,i+k+2}) = \phi(T_{m,i+1,k-i-2})[\phi(T_{m,i+1,k-i-1}) - \phi(T_{m,i,k-i-2})]. \tag{9}$$

Since the second factors in (8) and (9) are the same, we are done (same argumentation as above).

This completes the proof. □

The following function $f_{m,k}(\lambda)$ is defined on the basis of the proof of Theorem 3.1:

$$\begin{aligned}f_{m,k}(\lambda) &= \phi(T_{m,i,k-i}) - \phi(T_{m,i-1,k-i-1}) = \phi(T_{m,1,k-1}) - \phi(T_{m,0,k-2}) \\ &= \phi(T_{m,1,k-1}) - \phi(P_{m+k-1}) = \phi(P_{m+k+1}) - \phi(P_m)\phi(P_{k-1}),\end{aligned}$$

which implies that the largest root of $f_{m,k}(\lambda)$ is the index of the graph $P^{2k}_{i,i+k+1}$ for $0 \leqslant i \leqslant \frac{k-1}{2}$.

Remark 3.1. Since P_m and P_{k-1} are proper subgraphs of P_{m+k+1}, then $\rho(P_m \cup P_{k-1}) < \rho(P_{m+k+1}) < 2$. Let $\theta_{m,k}$ be the largest root of $f_{m,k}(\lambda)$, and assume that $\theta_{m,k} > 2$. Then for any $\lambda \in (2, \theta_{m,k})$ we get that $f_{m,k}(\lambda) < 0$.

Lemma 3.1. *Let* $1 \leqslant i < \lfloor\frac{n-k-1}{2}\rfloor$. *Then*

$$\phi(P^n_{i,i+k+1}) - \phi(P^n_{i+1,i+k+2}) = \phi(P^{n-2i}_{0,k+1}) - \phi(P^{n-2i}_{1,k+2}).$$

Proof. We will prove the lemma by induction on i. Set $i = 1$, then $n - 2 \geqslant k + 2$. By applying Corollary 2.1 to $P^n_{1,k+2}$ at the pendant edge positioned on the right side, we have

$$\phi(P^n_{1,k+2}) = \lambda\phi(P^{n-1}_{1,k+2}) - \phi(P^{n-2}_{1,k+2}).$$

By applying Corollary 2.1 to $P^n_{2,k+3}$ at the pendant edge positioned on the left side, we get

$$\phi(P^n_{2,k+3}) = \lambda\phi(P^{n-1}_{1,k+2}) - \phi(P^{n-2}_{0,k+1}).$$

Therefrom

$$\phi(P^n_{1,k+2}) - \phi(P^n_{2,k+3}) = \phi(P^{n-2}_{0,k+1}) - \phi(P^{n-2}_{1,k+2}).$$

We now assume that $i \geqslant 2$, and that the result of the lemma holds for $i-1$. That is,

$$\phi(P^n_{i-1,i+k}) - \phi(P^n_{i,i+k+1}) = \phi(P^{n-2(i-1)}_{0,k+1}) - \phi(P^{n-2(i-1)}_{1,k+2}).$$

Consider next the graphs $P^n_{i,i+k+1}$ and $P^n_{i+1,i+k+2}$. By similar calculations and inductive assumption, we have

$$\phi(P^n_{i,i+k+1}) - \phi(P^n_{i+1,i+k+2}) = \phi(P^{n-2}_{i-1,i+k}) - \phi(P^{n-2}_{i,i+k+1}) = \phi(P^{n-2i}_{0,k+1}) - \phi(P^{n-2i}_{1,k+2}).$$

This completes the proof. $\square$

We now discuss the index of the graphs $P^{2,2,n}_{0,k+1}$ and $P^{2,2,n}_{1,k+2}$ for $n = k+3$. Note that $P^{2,2,k+3}_{0,k+1} \cong T(2,2,k+3)$ and $P^{2,2,k+3}_{1,k+2} \cong P^{1,1,k+5}_{2,k+3}$. Suppose that $n < 2k$ (and thus $k \geqslant 4$). So by [3,4] we arrive at $2 < \rho(P^{2,2,k+3}_{0,k+1}), \rho(P^{2,2,k+3}_{1,k+2}) < \sqrt{2+\sqrt{5}}$.

Lemma 3.2. *If $n = k+3$ and $n < 2k$ (or $k \geqslant 4$), then $\rho(P^{2,2,k+3}_{1,k+2}) < \rho(P^{2,2,k+3}_{0,k+1})$.*

Proof. By Lemma 2.4 we get

$$\rho(P^{2,2,k+3}_{1,k+2}) < \rho(P^{2,2,k+2}_{1,k+1}) \leqslant \rho(P^{2,2,6}_{1,5}) \quad \text{and} \quad \rho(P^{2,2,7}_{0,5}) \leqslant \rho(P^{2,2,k+2}_{0,k}) < \rho(P^{2,2,k+3}_{0,k+1}),$$

where in the second chain of inequalities k is greater than 4. By software Mathematica, we have that $\rho(P^{2,2,6}_{1,5}) = 2.05288\cdots < \rho(P^{2,2,7}_{0,5}) = 2.05503\cdots$, and the proof follows. $\square$

Theorem 3.2. *Let $0 \leqslant i < j \leqslant \lfloor\frac{n-k-1}{2}\rfloor$. Then*

(i) *if $n > 2k$, then $\rho(P^n_{i,i+k+1}) < \rho(P^n_{j,j+k+1})$;*
(ii) *if $n < 2k$, then $\rho(P^n_{i,i+k+1}) > \rho(P^n_{j,j+k+1})$.*

Proof. Without loss of generality, let $j = i+1$. So $i+1 \leqslant \lfloor\frac{n-k-1}{2}\rfloor$. Set $l = n-2i-k-3$. Obviously, $l \geqslant 0$. By Lemma 3.1, since $n-2i = l+k+3$ we get

$$\begin{aligned}\Omega &= \phi(P^n_{i,i+k+1}) - \phi(P^n_{i+1,i+k+2}) = \phi(P^{n-2i}_{0,k+1}) - \phi(P^{n-2i}_{1,k+2}) \\ &= \phi(P^{l+k+3}_{0,k+1}) - \phi(P^{l+k+3}_{1,k+2}).\end{aligned} \tag{10}$$

We first suppose that $l = 0$. Then write

$$\phi(P^{k+3}_{0,k+1}) - \phi(P^{k+3}_{1,k+2}) = g_{m,k}(\lambda). \tag{11}$$

We next suppose $l \geqslant 1$ (so $n \geqslant k+4$). Applying Corollary 2.1 at the vertex $k+4$ of both $P^{l+k+3}_{0,k+1}$ and $P^{l+k+3}_{1,k+2}$, we arrive at

$$\phi(P_{0,k+1}^{l+k+3}) = \lambda\phi(P_{0,k+1}^{k+3})\phi(P_{l-1}) - \phi(T_{m,1,m+k+1})\phi(P_{l-1}) - \phi(P_{0,k+1}^{k+3})\phi(P_{l-2}) \tag{12}$$

and

$$\phi(P_{1,k+2}^{l+k+3}) = \lambda\phi(P_{1,k+2}^{k+3})\phi(P_{l-1}) - \phi(T_{m,1,m+k+1})\phi(P_{l-1}) - \phi(P_{1,k+2}^{k+3})\phi(P_{l-2}) \tag{13}$$

(Note, we set $\phi(P_{-1}) = 0$. Namely, since $\phi(P_n) = \lambda\phi(P_{n-1}) - \phi(P_{n-2})$ for $n \geqslant 2$ and $\phi(P_0) = 1$, we have $\phi(P_1) = \lambda\phi(P_0) - \phi(P_{-1})$, and so $\phi(P_{-1}) = 0$.)

From equalities (10), (12) and (13), it follows that

$$\Omega = (\lambda\phi(P_{l-1}) - \phi(P_{l-2}))(\phi(P_{0,k+1}^{k+3}) - \phi(P_{1,k+2}^{k+3})) = \phi(P_l)g_{m,k}(\lambda).$$

Note that P_l is a proper subgraph of the graphs $P_{i,i+k+1}^n$ and $P_{i+1,i+k+2}^n$, and that, for $\lambda \geqslant \min\{\rho(P_{i,i+k+1}^n), \rho(P_{i+1,i+k+2}^n)\}$, Ω and $g_{m,k}(\lambda)$ have the same sign, since $\phi(P_l) = \phi(P_l, \lambda) > 0$.

We now show that $g_{m,k}(\lambda) = \phi(P_{m-1})f_{m,k}(\lambda)$. By applying Corollary 2.1 repeatedly, we obtain from (11) that

$$\begin{aligned}
g_{m,k}(\lambda) &= \phi(P_{0,k+1}^{k+3}) - \phi(P_{1,k+2}^{k+3}) \\
&= [\lambda\phi(T_{m,1,m+k+1}) - \phi(P_{2m+k+2})] - [\lambda\phi(T_{m,1,m+k+1}) - \phi(P_m)\phi(T_{m,1,k})] \\
&= \phi(P_m)\phi(T_{m,1,k}) - \phi(P_{2m+k+2}) \\
&= \phi(P_m)[\lambda\phi(P_{m+k+1}) - \phi(P_m)\phi(P_k)] \\
&\quad - [\phi(P_{m+k+1})\phi(P_{m+1}) - \phi(P_{m+k})\phi(P_m)] \\
&= \phi(P_{m+k+1})[\lambda\phi(P_m) - \phi(P_{m+1})] + \phi(P_m)[\phi(P_{m+k}) - \phi(P_m)\phi(P_k)] \\
&= \phi(P_{m+k+1})\phi(P_{m-1}) + \phi(P_m)[\phi(P_m)\phi(P_k) \\
&\quad - \phi(P_{m-1})\phi(P_{k-1}) - \phi(P_m)\phi(P_k)] \\
&= \phi(P_{m-1})[\phi(P_{m+k+1}) - \phi(P_m)\phi(P_{k-1})] = \phi(P_{m-1})f_{m,k}(\lambda).
\end{aligned} \tag{14}$$

Thus $\theta_{m,k}$ is also the largest root of $g_{m,k}(\lambda)$. Since $\lim_{\lambda\to+\infty} g_{m,k}(\lambda) = +\infty$, we get that $g_{m,k}(\lambda) > 0$ for $\lambda > \theta_{m,k}$.

Finally, we can prove the theorem.

(i) Let any graph $G \in \mathscr{G}_{i,j}^n$ such that $l(G) > 2k$ (see Remark 1.1). We delete some vertices from the main path of G in order to obtain a graph $H \in \mathscr{G}_{i',j'}^{n'}$ with $L(H) = 2k$. Then the spectral radius of H is exactly $\theta_{m,k}$, and it is the same for any choice that we can do for deletion of the vertices in the main path. So $\theta_{m,k} < \rho(G)$ for any graph in question. Therefore $\phi(P_{i,i+k+1}^n) - \phi(P_{i+1,i+k+2}^n) > 0$ for any $\lambda > \theta_{m,k}$ and, by (i) of Lemma 2.5, we get $\rho(P_{i,i+k+1}^n) < \rho(P_{i+1,i+k+2}^n)$, as required.

(ii) Let any graph $G \in \mathscr{G}_{i,j}^n$ such that $l(G) < 2k$. We add some vertices and edges to the main path of G in order to obtain a graph $H \in \mathscr{G}_{i'',j''}^{n''}$ with $L(H) = 2k$. Then the spectral radius of H is exactly $\theta_{m,k}$, and it is the same for any choice that we can do for the adding of the vertices in the main path. So $\theta_{m,k} > \rho(G)$ for any graph in question. The following cases should be discussed:

Case 1. $\rho(G) \leqslant 2$ or $2 < \rho(G) < \sqrt{2+\sqrt{5}}$.

In this case, the theorem has been proved except $G = P_{0,k+1}^{2,2,k+3}$ (see [1, Theorem 3.4]). By Lemma 3.2 the theorem still holds for this exception.

Case 2. $\sqrt{2+\sqrt{5}} < \rho(G) < \theta_{m,k}$.

In this case, there exist real numbers a and b such that

$$\sqrt{2+\sqrt{5}} < a < \min\{\rho(P^n_{i,i+k+1}), \rho(P^n_{i+1,i+k+2})\} \\ < \max\{\rho(P^n_{i,i+k+1}), \rho(P^n_{i+1,i+k+2})\} < b < \theta_{m,k}.$$

Note that $\phi(P_{m-1}) > 0$ for any $\lambda \in (a, b)$ (since $\rho(P_{m-1}) < 2$). Recalling Remark 3.1, we obtain from (7) that $g_{m,k}(\lambda) < 0$ for $\lambda \in (a, b)$ which results in $\phi(P^n_{i,i+k+1}) - \phi(P^n_{i+1,i+k+2}) < 0$. By (ii) of Lemma 2.5, it follows that $\rho(P^n_{i,i+k+1}) > \rho(P^n_{i+1,i+k+2})$. □

Theorem 3.3. *For fixed m, let $\theta_{m,k}$ be the largest root of $f_{m,k}(\lambda)$. Then $\{\theta_{m,k}\}_{k\in\mathbb{N}}$ is an increasing sequence and*

$$\lim_{k\to+\infty} \theta_{m,k} \begin{cases} = \sqrt{2+\sqrt{5}} & if\ m = 1; \\ < \frac{3\sqrt{2}}{2} & if\ m \geqslant 2. \end{cases}$$

Proof. For the case that $m = 1$, the theorem has been proved (see [1, Theorem 3.5]). We now show the case $m \geqslant 2$. From Theorem 3.1 and Remark 3.1, it follows that $\theta_{m,k}$ is the index of the graph $P^{2k}_{0,k+1} = T_{m,k-1,m+k+1}$, and $\theta_{m,k+1}$ is the index of the graph $P^{2(k+1)}_{0,k+2} = T_{m,k,m+k+2}$. Since $T_{m,k-1,m+k+1} \subset T_{m,k,m+k+2}$, then $\theta_{m,k} < \theta_{m,k+1}$.

Since $T_{m,k-1,m+k+1} \subset T_{m,k+1,m+k+1} \subset T_{m,k+1,m+k+3} = P^{2(k+2)}_{0,k+3}$, then we get that $\theta_{m,k} < \rho(T_{m,k+1,m+k+1}) < \theta_{m,k+3}$ for any $k \in \mathbb{N}$. Thus $\lim_{k\to+\infty} \theta_{m,k} = \lim_{r\to+\infty} \rho(T_{m,r,r}) < \lim_{r\to+\infty} \rho(T_{r,r,r})$. Since $\lim_{r\to+\infty} \rho(T_{r,r,r}) = \frac{3\sqrt{2}}{2}$ (see [9]), the theorem holds. □

Remark 3.2. It is obvious that the results of Theorems 3.1–3.3 are independent on the diameter d of the graph $P^n_{i,j}$; for $m = 1$ see [1].

Now we give an upper bound for the spectral radius of the graphs $P^{s,t,n}_{i,j}$. Let us denote $P^{r,r,2r+1}_{r,r+1} = G(r)$.

Lemma 3.3. *For any s, t and n we have $\rho(P_{s+t+n+1}) \leqslant \rho(P^{s,t,n}_{i,j}) \leqslant \rho(G(r))$, where $r \geqslant \max\{s, t, i, n-j\}$.*

Proof. The proof easily follows from Lemma 2.4. □

Theorem 3.4. *For any r we have*

$$\rho(G(r)) < \sqrt{5}.$$

Proof. We prove that $\lim_{r\to+\infty} \rho(G(r)) = \sqrt{5}$.

Consider the Perron eigenvector $\mathbf{x}$ of $G(r)$. Due to the symmetry x_u and x_v have the same weight, and any vertex indicated in Fig. 3 by i, $1 \leqslant i \leqslant r$, have also the same weight.

The following relations come from the eigenvalue equation:

$$\rho x_r = x_{r-1}$$

$$\rho x_{r-1} = x_{r-2} + x_r$$

$$\cdots$$

$$\rho x_1 = x_2 + x_u$$

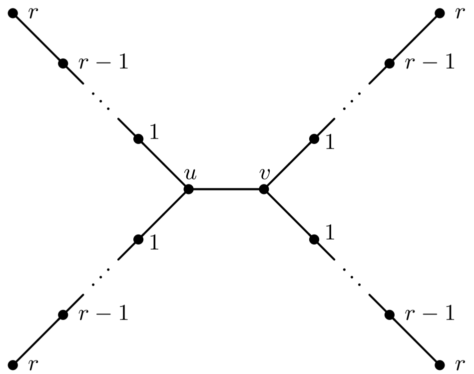

Fig. 3. The graph $G(r)$.

Then we have $x_1 = \frac{ax_r + cx_u}{b}$ and $x_{r-1} = \frac{cx_r + ax_u}{b}$ where $a = \sinh t$, $b = \sinh(rt)$, $c = \sinh((r-1)t)$ and $t = \ln \frac{\rho + \sqrt{\rho^2 - 4}}{2}$. By simple calculations it is possible to have the weight of the vertices expressed in function of x_u, in particular $x_1 = (\frac{c}{b} + \frac{a^2}{b(b\rho - c)})x_u$. Applying (1) at vertex u we have $\rho x_u = x_v + 2x_1$ and, since $x_u = x_v$, we have

$$\rho x_u = x_u + 2\left(\frac{c}{b} + \frac{a^2}{b(b\rho - c)}\right)x_u.$$

Then $\rho(G(r))$ must satisfy the following equation:

$$(\rho - 1) = 2\left(\frac{c}{b} + \frac{a^2}{b(b\rho - c)}\right).$$

By making $r \to +\infty$ then $\rho(G(r))$ increases, $\frac{a^2}{b(b\rho - c)} \to 0$ and $\frac{c}{b} \to \frac{2}{\rho + \sqrt{\rho^2 - 4}}$. So $\rho(G(r))$ is less than the largest root of the following equation:

$$\rho - 1 - \frac{4}{\rho + \sqrt{\rho^2 - 4}} = 0. \tag{15}$$

Since the largest root of (15) is $\sqrt{5}$ we are done. □

4. A result on unicyclic graphs

Let v_0 and v_k be two vertices of a cycle C_g with girth g such that $D(v_0, v_k) = k$, where $1 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$. Let $C_{s,t}^{g,k}$ (simply $C_{s,t}^{k}$) be an unicyclic graph obtained from C_g by identifying the vertex v_0 with a pendant vertex of path P_{s+1} and the vertex v_k with a pendant vertex of path P_{t+1}, respectively. Let $C_g(P_n)$ stand for the graph obtained from the cycle C_g and the path P_n by identifying a vertex of C_g and a pendant vertex of P_n.

By $\partial(f)$ and $\rho(f)$ we denote, respectively, degree and maximum root of the polynomial $f(x)$.

Lemma 4.1. *Let $f_1(x)$ and $f_2(x)$ be two polynomials in x with leading coefficients positive satisfying $\partial(f_2) > \partial(f_1)$ and $\rho(f_2) > \rho(f_1)$. If $f_3(x) = f_2(x) - f_1(x)$, then $f_3(x)$ has at least one root and $\rho(f_3) > \rho(f_2)$.*

Proof. Since $\partial(f_2) > \partial(f_1)$ and the leading coefficients of $f_1(x)$ and $f_3(x)$ are positive, then

$$f_1(x) \to +\infty \quad \text{and} \quad f_3(x) \to +\infty \quad \text{when } x \to +\infty.$$

By $\rho(f_2) > \rho(f_1)$, we arrive at $f_1(\rho(f_2)) > 0$, together with $f_3(\rho(f_2)) = -f_1(\rho(f_2))$ we have $f_3(\rho(f_2)) < 0$. Thus the graph of the continuous function $f_3(x)$ intersects the x-axis at $x > \rho(f_2)$, namely $\rho(f_3) > \rho(f_2)$. $\square$

Lemma 4.2. *Let r_i and s_i $(i = 1, 2)$ be non-negative integers such that $0 \leqslant r_1 \leqslant r_2$, $r_1 \leqslant s_1 \leqslant s_2$ and $r_1 + r_2 = s_1 + s_2$. Then*

$$\phi(P_{r_1})\phi(P_{r_2}) - \phi(P_{s_1})\phi(P_{s_2}) = -\phi(P_{s_1-r_1-1})\phi(P_{s_2-r_1-1}).$$

Proof. By Lemma 2.1 and $r_1 + r_2 = s_1 + s_2$ we get that

$$\begin{aligned}\phi(P_{r_1})\phi(P_{r_2}) - \phi(P_{r_1-1})\phi(P_{r_2-1}) &= \phi(P_{r_1+r_2}) = \phi(P_{s_1+s_2}) \\ &= \phi(P_{s_1})\phi(P_{s_2}) - \phi(P_{s_1-1})\phi(P_{s_2-1}),\end{aligned}$$

which leads to

$$\phi(P_{r_1})\phi(P_{r_2}) - \phi(P_{s_1})\phi(P_{s_2}) = \phi(P_{r_1-1})\phi(P_{r_2-1}) - \phi(P_{s_1-1})\phi(P_{s_2-1}). \tag{16}$$

By (16) and $\phi(P_0) = 1$ (see (13)) we arrive at

$$\begin{aligned}\phi(P_{r_1})\phi(P_{r_2}) - \phi(P_{s_1})\phi(P_{s_2}) &= \phi(P_{r_1-1})\phi(P_{r_2-1}) - \phi(P_{s_1-1})\phi(P_{s_2-1}) \\ &= \phi(P_{r_1-2})\phi(P_{r_2-2}) - \phi(P_{s_1-2})\phi(P_{s_2-2}) \\ &= \cdots \\ &= \phi(P_{r_1-r_1})\phi(P_{r_2-r_1}) - \phi(P_{s_1-r_1})\phi(P_{s_2-r_1}) \\ &= \phi(P_{r_2-r_1}) - \phi(P_{s_1-r_1})\phi(P_{s_2-r_1})\end{aligned} \tag{17}$$

Note $r_2 - r_1 = (s_1 - r_1) + (s_2 - r_1)$. So we obtain from Lemma 2.1 that

$$\phi(P_{r_2-r_1}) = \phi(P_{s_1-r_1})\phi(P_{s_2-r_1}) - \phi(P_{s_1-r_1-1})\phi(P_{s_2-r_1-1}) \tag{18}$$

and the result follows from (17) and (18). $\square$

Theorem 4.1. *Let $2 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$, then $\rho(C_{s,t}^k) < \rho(C_{s,t}^{k-1})$.*

Proof. Let the edge $e = uv$ belong to the path of length $t + 1$ in $C_{s,t}^{k-1}$ and $C_{s,t}^k$, respectively, such that $d(u) = 3$ and $d(v) = 2$. By applying Corollary 2.1 to the edge e, we get

$$\phi(C_{s,t}^{k-1}) = \phi(P_t)\phi(C_g(P_{s+1})) - \phi(P_{t-1})\phi(T_{s,k-2,g-k}).$$
$$\phi(C_{s,t}^{k}) = \phi(P_t)\phi(C_g(P_{s+1})) - \phi(P_{t-1})\phi(T_{s,k-1,g-k-1}).$$

Similarly, we arrive at

$$\phi(T_{s,k-2,g-k}) = \phi(P_s)\phi(P_{g-1}) - \phi(P_{s-1})\phi(P_{k-2})\phi(P_{g-k})$$
$$\phi(T_{s,k-1,g-k-1}) = \phi(P_s)\phi(P_{g-1}) - \phi(P_{s-1})\phi(P_{k-1})\phi(P_{g-k-1}).$$

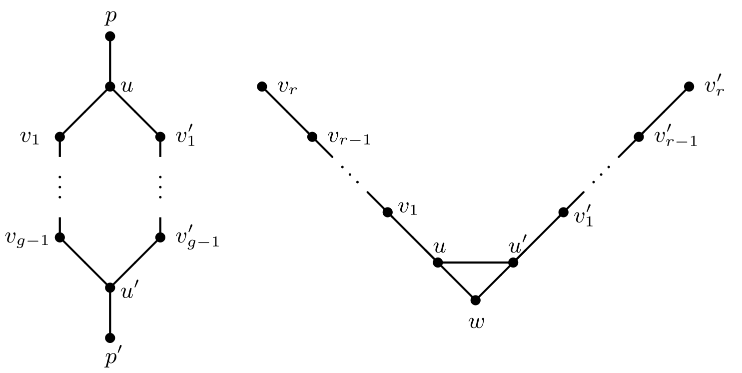

Fig. 4. The graphs $C_{1,1}^{2g,g}$ and $C_{r,r}^{3,1}$.

From the above equalities and Lemma 4.2, it follows that

$$\begin{aligned}\phi(C_{s,t}^{k-1}) - \phi(C_{s,t}^{k}) &= \phi(P_{s-1})\phi(P_{t-1})[\phi(P_{k-2})\phi(P_{g-k}) - \phi(P_{k-1})\phi(P_{g-k-1})] \\ &= -\phi(P_{s-1})\phi(P_{t-1})\phi(P_{g-2k}) = -h(\lambda).\end{aligned}$$

Note that $\partial(\phi(C_{s,t}^{k})) > \partial(h)$ and $\rho(C_{s,t}^{k}) > \rho(h)$ (since $P_{s-1}, P_{t-1}, P_{g-2k} \subset C_{s,t}^{k}$). Then by Lemma 4.1 we obtain $\rho(C_{s,t}^{k-1}) > \rho(C_{s,t}^{k})$. $\square$

Remark 4.1. The special case $s = t = 1$ of the above theorem is also proved in [1].

Finally, in a similar way as done in Section 3, we give some bounds for the spectral radius of $C_{s,t}^{g,k}$. We first consider the lemma below whose proof can be easily deduced by Lemma 2.4:

Lemma 4.3. *For any k, g, t and s, $\rho(C_{1,1}^{2g,g}) < \rho(C_{s,t}^{g,k}) \leqslant \rho(C_{r,r}^{3,1})$, where $r \geqslant \max\{s, t\}$.*

Now we give a lower bound for the spectral radius of the graphs $C_{1,1}^{2g,g}$ and an upper bound for the graphs $C_{r,r}^{3,1}$.

Theorem 4.2. *The following inequalities hold for any $g \geqslant 3$ and $r \geqslant 1$:*

(i) $\sqrt{2+\sqrt{5}} < \rho(C_{1,1}^{2g,g})$,
(ii) $\rho(C_{r,r}^{3,1}) < \alpha = \frac{\sqrt[3]{54-6\sqrt{33}}}{3} + \frac{\sqrt[3]{54+6\sqrt{33}}}{3} \approx 2.382975.$

Proof. Let us consider the inequality (i), and let $\mathbf{x}$ be the Perron eigenvector of $C_{1,1}^{2g,g}$.

Let us label the graph in this way: $u(= v_0 = v_0')$ and $u'(= v_g = v_g')$ are the vertices of degree 3, p and p' their pendant vertex respectively, $v_1, \ldots, v_g$ the vertices going from u to u' on one part of the cycle and $v_1', \ldots, v_g'$ be the vertices going from u to u' on the other part of the cycle (see Fig. 4). By symmetry we have the following equalities for the component of the Perron eigenvector: $x_{v_i} = x_{v_i'}$ and $x_{v_i} = x_{v_{g-i}}$, for any $0 \leqslant i \leqslant g$. The following relations come from the eigenvalue equation:

$$\rho x_{v_1} = x_{v_2} + x_u$$
$$\rho x_{v_2} = x_{v_3} + x_{v_1}$$
$$\ldots$$
$$\rho x_{v_{g-1}} = x_{v_{g-2}} + x_{u'}$$

Then we have $x_{v_1} = \frac{ax_{u'}+cx_u}{b} = \frac{ax_u+cx_u}{b} = x_{v_{g-1}}$, where $a = \sinh t$, $b = \sinh(gt)$, $c = \sinh((g-1)t)$ and $t = \ln \frac{\rho+\sqrt{\rho^2-4}}{2}$. By considering (1) in u then we get

$$\rho x_u = x_p + x_{v_1} + x_{v_1'} = \frac{\rho}{x_u} + 2x_{v_1}. \tag{19}$$

Then the spectral radius of $C_{1,1}^{2g,g}$ must satisfy the following equation:

$$\rho = \frac{1}{\rho} + 2\frac{a+c}{b}.$$

If we make $g \to +\infty$ (so the girth of the graph increases while its spectral radius decreases) then $\frac{a}{b} \to 0$ and $\frac{c}{b} \to \frac{2}{\rho+\sqrt{\rho^2-4}}$ we obtain that $\rho(C_{1,1}^{2g,g})$ is larger than the largest root of the following equation:

$$\rho - \frac{1}{\rho} - \frac{4}{\rho + \sqrt{\rho^2 - 4}} = 0.$$

The largest positive root of the previous equation is $\rho = \sqrt{2+\sqrt{5}}$ and (i) is proved.

In order to prove the inequality (ii) we follow a similar routine. Consider the graph $C_{r,r}^{3,1}$, its Perron eigenvector $\mathbf{x}$ and the following labeling for its vertices: let w be the unique vertex of degree 2 in the cycle, u and u' the two vertices of degree 3 and $v_1, \ldots, v_r$ ($v_1', \ldots, v_r'$) the vertices of the paths going from the vertex u (resp. u'). By considering the eigenvalue equations (analogously to the inequality (i)) and by making $r \to +\infty$ (note that in this case the spectral radius increases) we get that $\rho(C_{r,r}^{3,1})$ is less than the largest root of the following equation:

$$\rho - \frac{2}{\rho} - 1 - \frac{2}{\rho + \sqrt{\rho^2 - 4}} = 0.$$

By simple calculations we get finally get that the largest root is $\alpha = \frac{\sqrt[3]{54-6\sqrt{33}}}{3} + \frac{\sqrt[3]{54+6\sqrt{33}}}{3} \approx 2.382975$. This completes the proof. $\square$

Corollary 4.1. *Consider $C_{s,t}^{g,k}$, where $g \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{g}{2} \rfloor$. Then, for any positive integers s and t, the following inequalities hold*:

$$\sqrt{2+\sqrt{5}} < \rho(C_{s,t}^{g,k}) < \alpha \quad (\approx 2.382975)$$

Proof. From Lemma 2.4 and Theorem 4.1(i), it follows that $\rho(C_{s,t}^{g,k}) > \rho(C_{s,t}^{2g,k}) \geqslant \rho(C_{s,t}^{2g,g})$. Since $C_{1,1}^{2g,g}$ is a subgraph of $C_{s,t}^{2g,g}$, from Lemma 2.3 and Theorem 4.2(i) we get $\rho(C_{s,t}^{2g,g}) \geqslant \rho(C_{1,1}^{2g,g}) = \sqrt{2+\sqrt{5}}$ which leads to $\rho(C_{s,t}^{g,k}) > \sqrt{2+\sqrt{5}}$. Similarly, by Lemma 2.4 and Theorem 4.1(ii) we get $\rho(C_{s,t}^{g,k}) \leqslant \rho(C_{r,r}^{3,1}) < \alpha$, where $r = \max(s,t)$.

This ends the proof. $\square$

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